Indian Statistical Institute, Bangalore
B. Math (Hons.) Third Year
Second Semester - Analysis IV

Back Paper Exam — Date: June 13, 2018
Maximum marks: 100 — Duration: 3 hours

Each question carries 10 marks

1. Let X be a compact metric space and $\mathcal{A} \subset C_{\mathbb{R}}(X)$ be a subalgebra that separates points of X and nowhere vanishes on X. Prove that $\mathcal{A}$ is dense in $C_{\mathbb{R}}(X)$.

2. (a) Let $\Phi\colon C[0,1] \to C[0,1]$ be given by $\Phi(f)(x) = \int_0^x f(t)dt$. Prove that Φ is Lipschitz. Can Lipschitz constant of Φ be less than one? (**Marks: 4**).

   (b) Let X be a complete metric space and $\phi\colon X \to X$ be a map such that $d(\phi^n(x), \phi^n(y)) \leq a_n d(x,y)$ for all $n \geq 1$ and all $x, y \in X$ for some sequence $a_n \to 0$. Prove that ϕ has a unique fixed point $x \in X$ and $\lim_{n\to\infty} \phi^n(y) = x$ for all $y \in X$.

3. (a) Prove that set of all invertible linear maps of $\mathbb{R}^n$ is open.

   (b) Prove that $\sum_1^\infty \frac{\sin nx}{n} = \frac{\pi - x}{2}$ for $0 < x < \pi$ (**Marks: 6**).

4. (a) Let $f \in \mathcal{R}[-\pi, \pi]$ be a 2π-periodic function and c_n be the Fourier coefficients of f. Suppose $f \in C^1(\mathbb{R})$. Prove that $\sum |c_n| < \infty$ and Fourier series of f converges uniformly.

   (b) Find the Fourier series of 2π-periodic function f given by $f(x) = x(\pi - |x|)$ on $(-\pi, \pi)$ (**Marks: 5**).

5. (a) Let $f \in \mathcal{R}[a.b]$. Prove that $\lim_{r\to\infty} \int_a^b f(t)\sin(rt+s)dt = 0$ for any $s \in \mathbb{R}$.

   (b) Using Fourier series prove that $\sum_{k=1}^\infty \frac{1}{k^6} = \frac{\pi^6}{945}$(**Marks: 5**).